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The Propagation of Infinitesimal Disturbances in an Ultrarelativistic Gas According to the Method of Elementary Solutions

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It has recently been shown that a linearized relativistic BGK model can be reduced, in the ultrarelativistic limit, to a system of three uncoupled transport equations for thermal, sound, and shear waves. The equation describing the propagation of thermal waves is the well-known one-speed neutron transport with isotropic scattering in the conservative case. In this paper the solution of the half-space problem for the equation describing the propagation of shear and sound waves is given according to Case's elementary solutions method.

KEY WORDS: Relativistic kinetic theory; transport theory.

1. INTRODUCTION

The propagation of small-amplitude disturbances in a relativistic gas has been the subject of a number of recent investigations.[1-3] In particular, it has been shown in Ref. 1 that, according to relativistic kinetic theory, signals propagate at a speed less than the speed of light. Relativistic generalizations of the classical BGK model[4] made an analytical treatment of the propagation of small-amplitude disturbances possible in complete analogy with the classical case.[5] The analysis of the dispersion relation for thermal, shear, and sound waves in a relativistic gas has been presented in Ref. 2, where the gas was assumed to behave according to the kinetic model first proposed by Anderson and Witting.[6] The one-dimensional, unsteady, linearized version of the same model has been considered in Ref. 3, where the proof is given that in the ultrarelativistic limit the model reduces to the

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following set of three uncoupled transport equations for thermal, shear, and sound waves, respectively:

$$\frac{\partial Y}{\partial t} + \mu \frac{\partial Y}{\partial x} + Y = \frac{1}{2} \int_{-1}^{+1} Y(x, t, \mu') \, d\mu' \tag{1a}$$

$$\frac{\partial W}{\partial t} + \mu \frac{\partial W}{\partial x} + W = \frac{3}{4} \int_{-1}^{+1} (1 - \mu'^2) \, W(x, t, \mu') \, d\mu' \tag{1b}$$

$$\frac{\partial Z}{\partial t} + \mu \frac{\partial Z}{\partial x} + Z = \frac{1}{2} \int_{-1}^{+1} Z(x, t, \mu') + \frac{3}{2} \mu \int_{-1}^{+1} \mu' Z(x, t, \mu') \, d\mu' \tag{1c}$$

In Eqs. (1), Y, W, and Z are functions related to the perturbation of the distribution function of the molecular velocities.(3) Equations (1) hold in the frame of reference at rest with respect to the unperturbed gas, x and t are normalized values of the space and time coordinate, respectively, and μ is the normalized x component of the velocity. The first equation of system (1) is the well-known one-speed neutron transport equation with isotropic scattering in the conservative case.(7) The method of elementary solutions was developed by Case(6) to solve the steady transport equation

$$\mu \frac{\partial Y}{\partial x} + Y = \frac{c}{2} \int_{-1}^{+1} Y(x, \mu') \, d\mu' \tag{2}$$

which becomes the steady version of Eqs. (1) if c is set equal to one. Case's method can be extended to solve unsteady transport equations by introducing the Laplace transform of the unknown $Y(x, t, \mu)$ with respect to the time variable and reducing the equation to a form similar to the steady one.(9,10) This technique is adopted in the present work to solve the half-space problem for Eqs. (1b) and (1c). It is worth noticing that the same set of transport equations occurs in a completely different context, in the study of small-amplitude signals propagation in a degenerate Fermi fluid.(11) This circumstance is not surprising, since, both in the case of an ultrarelativistic gas and in the case of a degenerate Fermi fluid, collisions affect only the velocity directions because all molecules have practically the same characteristic speed (the speed of light in the case of the relativistic gas and the Fermi speed in the gas of the degenerate Fermi gas).

2. ELEMENTARY SOLUTIONS OF THE SHEAR WAVE EQUATION

We shall start the analysis from Eq. (1b), because it turns out to be much simpler than Eq. (1c), providing a good starting point for a brief survey of the method.

The Laplace-transformed equation (1b) takes the form

$$(s+1)\hat{W}+\mu\frac{\partial\hat{W}}{\partial x}=\frac{3}{4}\int_{-1}^{+1}(1-\mu'^2)\,\hat{W}(x,s,\mu')\,d\mu' \tag{3}$$

In deriving Eq. (3), it has been assumed, without loss of generality, that $W(x, 0, \mu)$ vanishes. Solutions of Eq. (3) are now sought in the form

$$\hat{W}_s(x,\mu)=\exp[-(s+1)\,x/v]\,\phi_s(v\,|\,\mu) \tag{4}$$

Inserting this expression into Eq. (3), we obtain the following equation for the unknown $\phi_s(v\,|\,\mu)$:

$$\frac{s+1}{v}(v-\mu)\,\phi_s(v\,|\,\mu)=\frac{3}{4}\int_{-1}^{+1}(1-\mu'^2)\,\phi_s(v\,|\,\mu')\,d\mu' \tag{5}$$

Multiplying Eq. (5) by one and μ, respectively, and integrating the resulting equations provides two linear relationships for the moments of ϕ_s, by means of which Eq. (5) can be written as follows:

$$(v-\mu)\,\phi_s(v\,|\,\mu)=\frac{3}{2}\frac{v(1-v^2)}{2(s+1)-3v^2}\int_{-1}^{+1}\phi_s(v\,|\,\mu')\,d\mu' \tag{6}$$

for any finite value of v. If ϕ_s is chosen to have

$$\int_{-1}^{+1}\phi_s(v\,|\,\mu)\,d\mu=1 \tag{7}$$

we obtain

$$(v-\mu)\,\phi_s(v\,|\,\mu)=\frac{3}{2}\frac{v(1-v^2)}{2(s+1)-3v^2} \tag{8}$$

In solving Eq. (8), we have different answers, depending on whether or not v belongs to the interval $[-1, 1]$. When v is not an element of the interval, ϕ_s is the ordinary function

$$\phi_s(v\,|\,\mu)=\frac{3}{2}\frac{v(1-v^2)}{2(s+1)-3v^2}\frac{1}{v-\mu} \tag{9}$$

In Eq. (9), v and s are not independent. It is easily seen that, due to the normalization condition (7), they are related by the following relation:

$$\Omega(s,v)=\frac{\omega(s,v)}{2(s+1)-3v^2}=0 \tag{10}$$

where

$$\omega(s, v) = 2(s+1) - 3v^2 - 3v(1 - v^2)\,\mathrm{arctanh}(1/v) \tag{11}$$

It can be easily shown that $\lim_{v\to\infty} \omega(s, v) = 2s$; therefore, $\Omega(s, v)$ vanishes for any s fixed when v goes to infinity and when s and v are such that $\omega(s, v) = 0$. The eigenfunction corresponding to $v = \infty$ does not satisfy Eq. (5) in general; hence, it will not be considered. For any s fixed, the function $\omega(s, v)$ is an analytic function in the complex plane of the variable v cut along the interval $[-1, 1]$. When v belongs to the cut, Eq. (10) is satisfied by those values of s lying on the heart-shaped curve Γ shown in Fig. 1 and given by the equations

$$\mathrm{Re}(s^{\pm}) = \frac{3}{2} v^2 + \frac{3}{4} v(1 - v^2) \log \frac{1-v}{1+v} - 1 \tag{12a}$$

$$\mathrm{Im}(s^{\pm}) = \mp \frac{3}{4} \pi v(1 - v^2) \tag{12b}$$

the plus or minus sign applying when v tends to the cut from the upper or lower half-plane, respectively. The closed curve divides the complex plane

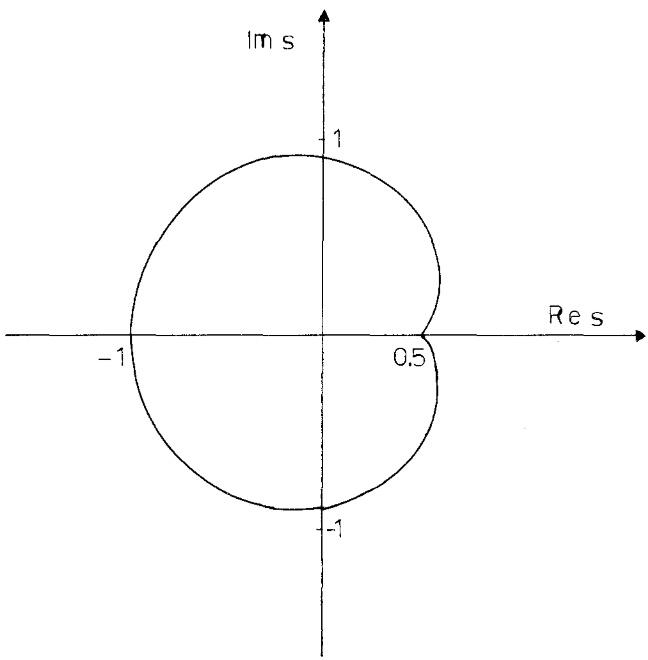

Fig. 1. The locus of complex s values that satisfy $\omega^{\pm}(s, v) = 0$.

of the variable s into two regions: R_1, the region enclosed by Γ, and $R_0 = C \backslash R_1$. The number N of zeros of the equation

$$\omega(s, v) = 0 \tag{13}$$

for s fixed can be established by using the argument principle:

$$\lim_{\substack{R \to \infty \\ \varepsilon \to 0}} \oint_{C = C_R + C_\varepsilon} dv \frac{d}{dv} \log \omega(s, v) = i\Delta_c \arg \omega = 2\pi i N \tag{14}$$

where C_R and C_ε are the contours shown in Fig. 2. Since $\omega(s, v)$ tends to a constant when v goes to infinity, the contribution of C_R vanishes when R tends to infinity; therefore we have

$$\lim_{\varepsilon \to 0} \oint_{C_\varepsilon} \frac{d}{dv} \log \omega \, dv = 2\pi i N \tag{15}$$

The function $\omega(s, v)$ is continuous and bounded in the vicinity of the points $v = \pm 1$; therefore, when ε tends to zero the contribution of the contours C_1 and C_3 vanishes as well and we obtain

$$N = \frac{1}{2\pi i} \left[\int_{-1}^{+1} \frac{d}{dv} \log \omega^+(s, v) \, dv - \int_{-1}^{+1} \frac{d}{dv} \log \omega^-(s, v) \, dv \right] \tag{16}$$

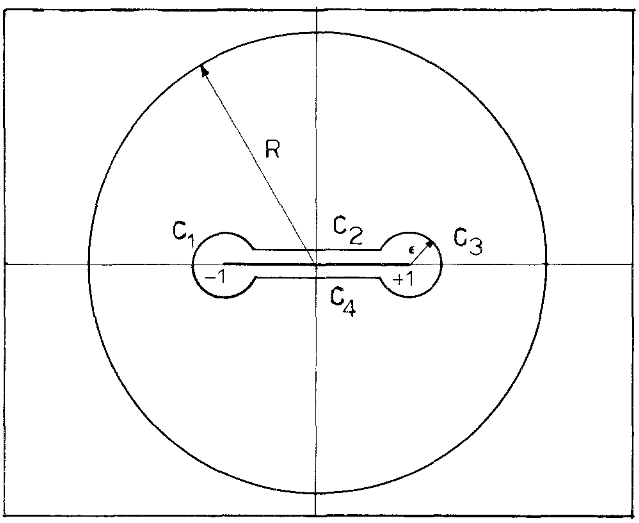

Fig. 2. Contour integration for the principle of the argument: $C_\varepsilon = C_1 + C_2 + C_3 + C_4$.

and finally

$$N=\frac{1}{2\pi i}\left[\log\frac{\omega^{+}(s,1)}{\omega^{-}(s,-1)}-\log\frac{\omega^{-}(s,1)}{\omega^{-}(s,-1)}\right] \tag{17}$$

It is evident from Eq. (18) that N is an even number, since we have

$$\omega^{+}(s,-v)=\omega^{-}(s,v) \tag{18}$$

Hence

$$N=\frac{1}{i\pi}\log\frac{\omega^{+}(s,1)}{\omega^{-}(s,1)} \tag{19}$$

This result will be useful later, but N will not be evaluated directly from Eq. (19). As shown by Eq. (16), we have

$$N=\frac{1}{2\pi}\Delta_{\text{cut}}\arg\omega(s,v) \tag{20}$$

When v belongs to the cut, $\omega(s,v)$ can be expressed as

$$\omega(s,v)=s-\bar{s}(v) \tag{21}$$

where $\bar{s}(v)$ belongs to Γ. Accordingly, the value of $\arg\omega(s,v)$ is given by the angle formed by the vector $s-\bar{s}(v)$ with the x axis. When v moves around the cut, $\bar{s}(v)$ moves counterclockwise along the curve and it is very simple to see that when s belongs to R_1 then $\Delta_{\text{cut}}\arg(s-\bar{s})=4\pi$. In this case Eq. (13) admits two opposite solutions, $v_0(s)$ and $-v_0(s)$. When s does not belong to the region surrounded by the curve, the variation of the argument of $s-\bar{s}(v)$ vanishes; therefore, Eq. (13) has no solutions in this case. The solution of Eq. (8) is no longer an ordinary function if v belongs to the interval $[-1,1]$, but it has to be interpreted as a distribution:

$$\psi_s(v\,|\,\mu)=\lambda_s(v)\,\delta(v-\mu)+\frac{3}{2}\frac{v(1-v^2)}{2(s+1)-3v^2}P\frac{1}{v-\mu} \tag{22}$$

the function λ_s has to be determined from the normalization condition (7). Straightforward calculation shows that

$$\lambda_s(v)=[\Omega^{+}(s,v)+\Omega^{-}(s,v)]/2 \tag{23}$$

where Ω^{+} and Ω^{-} are the limit values of the sectionally analytic function Ω when v tends to the cut from the upper or lower half-plane, respectively.

Because of its importance in applications, the proof of the half-space completeness of the eigenfunctions ϕ_s and ψ_s will now be given, following the standard method.[7,8] Let $f(\mu)$ be a complex function of the real variable μ , with $\mu \in [0, 1]$. It is assumed that f belongs to the set of functions that are Hölder continuous in the extended sense on the interval [0, 1]. It will be shown that $f(\mu)$ can be expressed as

$$f(\mu) = \int_0^1 A(\nu)\, \psi_s(\nu \mid \mu)\, d\nu \tag{24}$$

if $s \in R_0$, and

$$f(\mu) = a(\nu_0)\, \phi_s(\nu_0 \mid \mu) + \int_0^1 A(\nu)\, \psi_s(\nu \mid \mu)\, d\nu \tag{25}$$

if $s \in R_1$.

As is well known, the proof is by construction. Inserting expression (22) into Eq. (24), we obtain the following singular integral equation of the Cauchy type[12]:

$$f(\mu) = \lambda_s(\mu)\, A(\mu) + \frac{1}{i\pi} P \int_0^1 \frac{\Omega^+(s, \nu) - \Omega^-(s, \nu)}{2} \frac{A(\nu)}{\nu - \mu}\, d\nu \tag{26}$$

A sectionally analytic function $F(z)$ is now defined such that

$$F(z) = \frac{1}{2\pi i} \int_0^1 \frac{\Omega^+(s, \nu) - \Omega^-(s, \nu)}{2} \frac{A(\nu)}{\nu - z}\, d\nu \tag{27}$$

and taking into account the Plemelj formulas

$$F^+(\mu) - F^-(\mu) = \frac{\Omega^+(s, \nu) - \Omega^-(s, \nu)}{2} A(\mu) \tag{28a}$$

$$F^+(\mu) + F^+(\mu) = \frac{1}{i\pi} P \int_0^1 \frac{\Omega^+(s, \nu) - \Omega^-(s, \nu)}{2} \frac{A(\nu)}{\nu - \mu}\, d\nu \tag{28b}$$

we find that the singular integral equation (26) is reduced to the inhomogeneous Hilbert problem,

$$\Omega^+(s, \mu)\, F^+(\mu) - \Omega^-(s, \mu) F^- = \frac{1}{2} [\Omega^+(s, \mu) - \Omega^-(s, \mu)]\, f(\mu) \tag{29}$$

The function $F(z)$, whose limit values F^+ and F^- satisfy Eq. (29), can be expressed as

$$F(z) = X(z) \frac{1}{2\pi i} \int_0^1 \frac{1}{2} \frac{\Omega^+(s, \nu) - \Omega^-(s, \nu)}{\Omega^-(s, \nu)\, X^-(v)} \frac{f(v)}{v - z}\, d\nu \tag{30}$$

where $X(z)$ is a bounded, analytic function in the complex plane cut along the interval $[0, 1]$; furthermore, $X(z)$ is a solution of the homogeneous Hilbert problem,

$$\Omega^+ X^+ - \Omega^- X^- = 0 \tag{31}$$

$X(z)$ can be shown to be always bounded in the vicinity of $z = 0$ and to behave as $(1-z)^{-N/2}$ in the vicinity of $z = 1$. Accordingly, we must distinguish the following two cases:

1. When s belongs to R_0, $X(z)$ takes the form

$$X(z) = X_0(z) = \exp\left(- \frac{1}{2\pi i} \int_0^1 \log \frac{\omega^+}{\omega^-} \frac{1}{\nu - z} \, d\nu \right) \tag{32}$$

Taking into account Eqs. (28a) and (30), one finds the solution of the integral equation (26) to be

$$\begin{aligned} A(\mu) = {} & \frac{1}{2} \frac{\Omega^+(s, \mu) + \Omega^-(s, \mu)}{\Omega^+(s, \mu)\, \Omega^-(s, \mu)} f(u) \\ & - \frac{1}{2} \frac{X_0^-(s, \mu)}{\Omega^+(s, \mu)} \frac{1}{i\pi} \int_0^1 \frac{\Omega^+(s, \nu) - \Omega^-(s, \nu)}{\Omega^-(s, \nu)\, X_0^-(s, \nu)} \frac{f(\nu)}{\nu - \mu} \, d\nu \end{aligned} \tag{33}$$

This expression can be simplified if use is made of the following identities:

$$\frac{1}{X_0(z)} = 1 + \frac{1}{2\pi i} \int_0^1 \left(\frac{1}{X_0^+(\nu)} - \frac{1}{X_0^-(\nu)} \right) \frac{d\nu}{\nu - z} \tag{34a}$$

$$X_0(z)\, X_0(-z)\, \omega(s, z) = 2s \tag{34b}$$

$$\frac{1}{X_0(\mu)} = 1 + \frac{1}{2s} \frac{1}{2\pi i} \int_{-1}^0 \frac{[\omega^+(s, \nu) - \omega^-(s, \nu)]\, X_0(\nu)}{\nu + \mu} \, d\nu \tag{34c}$$

($\nu, \mu \in [-1, 0]$). The proof of identities (34) is standard.[7] It is worth noticing that the third identity is actually a nonsingular integral equation by means of which the values of $X_0(z)$ in the interval $[-1, 0]$ (and hence in the whole complex plane) can be calculated. Using Eq. (34b) to eliminate X_0 from Eq. (33), we obtain the following expression for the solution of Eq. (26):

$$\begin{aligned} A(\mu) = {} & \frac{2(s+1) - 3\nu^2}{2} \frac{1}{\omega^+ \omega^-} \Bigg[(\omega^+ + \omega^-) f(\mu) \\ & - \frac{1}{X_0(-\mu)} \frac{1}{i\pi} \int_0^1 \frac{(\omega^+ - \omega^-)\, X_0(-\nu)\, f(\nu)}{\nu - \mu} \, d\nu \Bigg] \end{aligned} \tag{35}$$

2. When s belongs to R_1 the function $X_0(z)$ is no longer bounded in the vicinity of $z=1$ and the proper function is

$$X_1(z)=(1-z)\exp\left(-\frac{1}{2\pi i}\int_0^1 \log\frac{\omega^+}{\omega^-}\frac{1}{v-z}\,dv\right) \tag{36}$$

In this case, however, the function $F(z)$ will not have the correct behavior at infinity unless the following condition holds:

$$\int_0^1 \frac{\omega^+-\omega^-}{\omega^- X_1^-} f(v)\,dv=0 \tag{37}$$

This is accomplished by setting

$$f(\mu)=g(\mu)-a(v_0)\,\phi_s(v_0\,|\,\mu) \tag{38}$$

where $a(v_0)$ is a coefficient to be determined, v_0 is the eigenvalue of the discrete spectrum saisfying Eq. (14) and such that $\mathrm{Re}[(s+1)/v_0]>0$, and $\phi_s(v_0\,|\,\mu)$ is the associated eigenfunction. The solution of Eq. (26) can now be written as

$$A(\mu)=\frac{1}{2}\frac{\Omega^++\Omega^-}{\Omega^+\Omega^-}f(\mu)-\frac{1}{2}\frac{X_1^-}{\Omega^+}\frac{1}{i\pi}\int_0^1\frac{(\Omega^+-\Omega^-)f(v)}{X_1^-\Omega^-}\frac{1}{v-\mu}\,dv \tag{39}$$

Since $\omega(s,z)$ has two roots, the identities have to be modified accordingly:

$$\frac{1}{X_1(z)}=\frac{1}{2\pi i}\int_0^1\left(\frac{1}{X_1^+(v)}-\frac{1}{X_1^-(v)}\right)\frac{1}{v-z}\,dv \tag{40a}$$

$$\frac{X_1(z)\,X_1(-z)\,\omega(s,z)}{v_0^2-z^2}=2s \tag{40b}$$

$$\frac{1}{X_1(\mu)}=\frac{1}{2\pi i}\frac{1}{2s}\int_{-1}^0\frac{(\omega^+-\omega^-)\,X_1(v)}{(v_0^2-v^2)(v+\mu)}\,dv, \qquad \mu, v\in[-1,0] \tag{40c}$$

In complete analogy with the pevious case, Eq. (40b) can be used to eliminate X_1^- from Eq. (39) and by means of partial fraction decomposition the following expression is obtained:

$$\begin{aligned}A(\mu)=\frac{2(s+1)-3\mu^2}{2}\frac{1}{\omega^+\omega^-}\Bigg\{&(\omega^++\omega^-)\,f(\mu)\\&+\frac{1}{X_1(-\mu)}\left[\frac{1}{i\pi}\int_0^1\frac{(\omega^+-\omega^-)\,X_1(-v)\,g(v)}{v-\mu}\,dv\right.\\&\left.-\frac{1}{i\pi}\int_0^1\frac{(\omega^+-\omega^-)\,X_1(-v)\,g(v)}{v-v_0}\,dv\right]+\frac{a(v_0)}{X_1(-\mu)}\,[I(v_0)-I(\mu)]\Bigg\}\end{aligned} \tag{41}$$

where

$$I(\mu)=\frac{1}{i\pi}\int_0^1 \frac{(\omega^+-\omega^-)\,X_1(-v)\,\phi_s(v_0\,|\,v)}{v-\mu}\,dv \tag{42a}$$

$$I(v_0)=\frac{1}{i\pi}\int_0^1 \frac{(\omega^+-\omega^-)\,X_1(-v)\,\phi_s(v_0\,|\,v)}{v-v_0}\,dv \tag{42b}$$

The coefficient $a(v_0)$ is to be evaluated from condition (38), which, taking into account the identity (32a), can be written as follows:

$$\int_0^1 \frac{(\omega^+-\omega^-)[g(v)-a(v_0)\,\phi_s(v_0\,|\,v)]\,X_1(-v)}{v_0^2-v^2}\,dv=0 \tag{43}$$

Since

$$\phi_s(v_0\,|\,v)=\frac{3}{2}\,\frac{v_0(1-v_0^2)}{2(s+1)-3v_0^2}\,\frac{1}{v_0-v}$$

we obtain

$$\frac{3}{2}\,\frac{v_0(1-v_0^2)}{2(s+1)-3v_0^2}\,a(v_0)\int_0^1 \frac{(\omega^+-\omega^-)\,X_1(-v)}{(v_0^2-v^2)(v_0-v)}\,dv$$
$$=\frac{1}{2v_0}\left[\int_0^1 \frac{(\omega^+-\omega^-)\,X_1(-v)\,g(v)}{v+v_0}\,dv-\int_0^1 \frac{(\omega^+-\omega^-)\,X_1(-v)\,g(v)}{v-v_0}\,dv\right] \tag{44}$$

The integral on the left-hand side of Eq. (44) is easily evaluated, since

$$\int_0^1 \frac{(\omega^+-\omega^-)\,X_1(-v)}{(v_0^2-v^2)(v_0-v)}\,dv$$
$$=-2s\int_0^1\left(\frac{1}{X_1^+}-\frac{1}{X_1^-}\right)\frac{1}{v-v_0}\,dv=-4\pi i\,\frac{s}{X_1(v_0)} \tag{45}$$

The integrals $I(\mu)$ and $I(v_0)$ are also easily calculated, taking into account identities (40a) and (40b), the second Plemelj formula for the sectionally analytic function $1/X_1(z)$, and the expansion of $1/X_1(z)$ for large z, to evaluate moments of the density $(1/X_1^+-1/X_1^-)$:

$$I(\mu)=\frac{3}{2}\,\frac{v_0(1-v_0^2)}{2(s+1)-3v_0^2}\left[2s+\frac{1}{2}\,\frac{(\omega^++\omega^-)\,X_1(-\mu)}{v_0-\mu}\right] \tag{46a}$$

$$I(v_0)=\frac{3}{2}\,\frac{sv_0(1-v_0^2)}{2(s+1)-3v_0^2}\left(\frac{1}{i\pi}\int_0^1 \log\frac{\omega^+}{\omega^-}\,dv-2v_0\right) \tag{46b}$$

3. THE PROPAGATION EQUATION FOR SOUND WAVES

The analysis of the half-space problem for the transport equation (1c) will now be given. As is to be expected, the calculations needed to obtain the coefficient of the eigenfunction expansion are considerably more involved.

The Laplace-transformed equation (1c) takes the form

$$(s+1)\hat{Z}+\mu\frac{\partial\hat{Z}}{\partial x}=\frac{1}{2}\int_{-1}^{+1}\hat{Z}(x,s,\mu')\,d\mu'+\frac{3}{2}\mu\int_{-1}^{+1}\mu'\hat{Z}(x,s,\mu')\,d\mu' \quad (47)$$

Equation (47) strongly resembles the steady neutron transport equation with linearly anisotropic scattering studied by Zelazny *et al.*[13] Separating variables as before, we obtain the following equation:

$$\frac{s+1}{\nu}(\nu-\mu)\,\phi_s(\nu\,|\,\mu)=\frac{1}{2}\int_{-1}^{+1}\phi_s(\nu\,|\,\mu')\,d\mu'+\frac{3}{2}\mu\int_{-1}^{+1}\mu'\phi_s(\nu\,|\,\mu')\,d\mu' \quad (48)$$

Integrating Eq. (48) with respect to μ gives

$$\int_{-1}^{+1}\mu\phi_s(\nu\,|\,\mu)\,d\mu=\nu\left(\frac{s}{s+1}\right)\int_{-1}^{+1}\phi_s(\nu\,|\,\mu')\,d\mu' \quad (49)$$

and substituting the above expression into Eq. (48), one obtains

$$(\nu-\mu)\,\phi_s(\nu\,|\,\mu)=\frac{1}{2}\frac{\nu}{s+1}\left(1+\frac{3\nu s}{s+1}\mu\right)\int_{-1}^{+1}\phi_s(\nu\,|\,\mu')\,d\mu' \quad (50)$$

Choosing normalized eigenfunctions ϕ_s, we find for the solution of Eq. (50)

$$\phi_s(\nu\,|\,\mu)=\frac{1}{2}\frac{\nu}{(s+1)}\left(1+\frac{3\nu^2 s}{s+1}\right)\frac{1}{\nu-\mu}-\frac{3}{2}\frac{\nu^2 s}{(s+1)^2} \quad (51)$$

when $\nu\notin[-1,1]$, and

$$\psi_s(\nu\,|\,\mu)=\lambda_s(\nu)\,\delta(\nu-\mu)+\frac{\Omega^+-\Omega^-}{2}P\frac{1}{i\pi}\frac{1}{\nu-\mu}-\frac{3}{2}\frac{s\nu^2}{(s+1)^2} \quad (52)$$

when $\nu\in[-1,1]$.

Let us consider first the eigenfunction of the discrete spectrum given by Eq. (51). The normalization condition implies that s and ν are related by the following relationship:

$$\Omega(s,\nu)=1+\frac{3\nu^2 s}{(s+1)^2}-\frac{\nu}{s+1}\left(1+\frac{3\nu^2 s}{s+1}\right)\operatorname{arctanh}\left(\frac{1}{\nu}\right)=0 \quad (53)$$

when v tends to the cut, the corresponding values of s satisfying Eq. (53) lie on the curve shown in Fig. 3 and given by the equation

$$s^2 + bs + c = 0 \tag{54}$$

where

$$c = 1 + \frac{v}{2}\left(\pm i\pi + \log\frac{1-v}{1+v}\right) \tag{55a}$$

$$b = 1 + (1 + 3v^2)c \tag{55b}$$

The curve divides the complex plane of the variable s into the regions R_0, R_1, and R_2. Following the same argument developed for the shear wave equation, it is easily shown that if s belongs to the outermost region R_0, Eq. (53) does not admit any solution; if s belongs to R_1, then Eq. (53) admits two opposite solutions v_0 and $-v_0$, and four solutions v_0, v_1, $-v_0$, and $-v_1$ when s belongs to the innermost region R_2.

The half-space completeness property of the set of eigenfunctions given by Eqs. (51) and (52) will now be estabilished, considering first the case in which $s \in R_0$: Starting from

$$f(\mu) = \int_0^1 A(v)\, \psi_s(v \mid \mu)\, dv \tag{56}$$

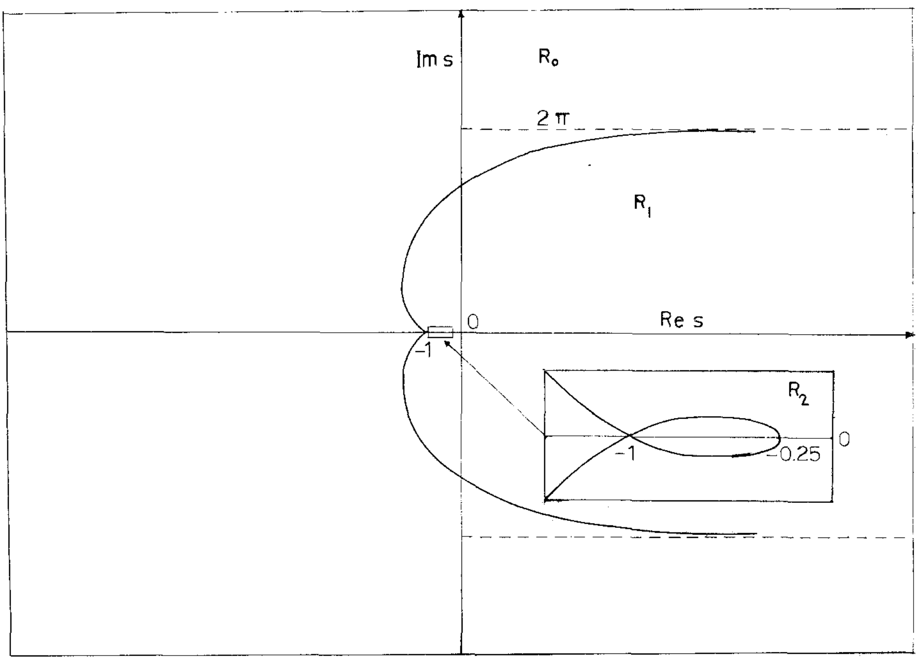

Fig. 3. The locus of complex s values that satisfy $\Omega^{\pm}(s, v) = 0$. Region R_2 is magnified.

and substituting the expression given by Eq. (2) into Eq. (56), we obtain the following singular integral equation:

$$f(\mu)=\lambda_s(\mu)\,A(\mu)+\frac{1}{i\pi}\int_0^1 \frac{\Omega^+-\Omega^-}{2}\,\frac{A(\nu)}{\nu-\mu}\,d\nu-\frac{3s}{(s+1)^2}\int_0^1 \nu^2 A(\nu)\,d\nu \tag{57}$$

Equation (57) differs from the corresponding equation of Section 2 because of the last integral on the right-hand side. If we set

$$k_s=\frac{3s}{(s+1)^2}\int_0^1 \nu^2 A(\nu)\,d\nu \tag{58a}$$

$$g(\mu)=f(\mu)+k_s \tag{58b}$$

Eq. (57) is brought into the form

$$g(\mu)=\lambda_s(\mu)\,A(\mu)+\frac{1}{i\pi}\int_0^1 \frac{(\Omega^+-\Omega^-)\,A(\nu)}{2}\,\frac{1}{\nu-\mu}\,d\nu \tag{59}$$

and solved leaving the constants k_s unspecified. After the solution of Eq. (58) has been found, k_s is determined from definition (58a). As shown in the previous section, the solution of Eq. (59) can be put into the form

$$A(\mu)=\frac{1}{\Omega^+\Omega^-}\left[\lambda_s(\mu)\,g(\mu)-\frac{1}{X_0(-\mu)}\,\frac{1}{i\pi}\int_0^1 \frac{1}{2}\,\frac{(\Omega^+-\Omega^-)\,X_0(-\nu)\,g(\nu)}{\nu-\mu}\,d\nu\right] \tag{60}$$

The function $X_0(z)$ is defined as follows:

$$X_0(z)=\exp\left(-\frac{1}{2\pi i}\int_0^1 \log\frac{\Omega^+}{\Omega^-}\,\frac{1}{\nu-z}\,d\nu\right) \tag{61}$$

The following identities hold for X_0:

$$\frac{1}{X_0(z)}=1+\frac{1}{2\pi i}\int_0^1\left(\frac{1}{X_0^+(\nu)}-\frac{1}{X_0^-(\nu)}\right)\frac{1}{\nu-z}\,d\nu \tag{62a}$$

$$X_0(z)\,X_0(-z)\,\Omega(s,z)=\left(\frac{s}{s+1}\right)^2 \tag{62b}$$

$$\frac{1}{X_0(\mu)}=1+\left(\frac{s+1}{s}\right)^2\frac{1}{2\pi i}\int_{-1}^0 \frac{(\Omega^+-\Omega^-)\,X_0(\nu)}{\nu+\mu}\,d\nu \tag{62c}$$

The value of the constant k_s is now determined by multiplying Eq. (60) by $3s/(s+1)^2\mu^2$ and integrating the resulting expression with respect to μ.

When $s \in R_1$ the coefficient $A(\mu)$ can be written as

$$A(\mu)=\frac{1}{\Omega^+\Omega^-}\left\{\lambda_s(\mu)\, g(\mu)-\frac{1}{X_1(-\mu)}\left[\frac{1}{i\pi}\int_0^1 \frac{1}{2}\frac{(\Omega^+-\Omega^-)\,X_1(-\nu)\,g(\nu)}{\nu-\mu}\,d\nu\right.\right.$$
$$\left.\left.-\frac{1}{i\pi}\int_0^1 \frac{1}{2}\frac{(\Omega^+-\Omega^-)\,X_1(-\nu)\,g(\nu)}{\nu-\nu_0}\,d\nu\right]\right\} \tag{63}$$

where

$$X_1(z)=(1-z)\exp\left(-\frac{1}{2\pi i}\int_0^1 \log\frac{\Omega^+}{\Omega^-}\frac{1}{\nu-z}\,d\nu\right)$$

In deriving Eq. (63) the identities

$$\frac{1}{X_1(z)}=\frac{1}{2\pi i}\int_0^1\left(\frac{1}{X_1^+(\nu)}-\frac{1}{X_1^-(\nu)}\right)\frac{d\nu}{\nu-z} \tag{64a}$$

$$\frac{X_1(z)\,X_1(-z)\,\Omega(s,z)}{(\nu_0^2-z^2)}=\left(\frac{s}{s+1}\right) \tag{64b}$$

$$\frac{1}{X_1(\mu)}=\frac{1}{2\pi i}\left(\frac{s+1}{s}\right)^2\int_{-1}^0 \frac{(\Omega^+-\Omega^-)\,X_1(\nu)}{(\nu_0^2-\nu^2)(\nu+\mu)}\,d\nu, \qquad \mu,\nu\in[-1,0] \tag{64c}$$

have been taken into account as well as the auxiliary condition

$$\int_0^1 \frac{(\Omega^+-\Omega^-)\,X_1(-\nu)\,g(\nu)}{\nu_0^2-\nu^2}\,d\nu=0 \tag{65}$$

which will be satisfied by adding an eigenfunction of the discrete spectrum to the expansion. Accordingly, we have so set

$$g(\mu)=f(\mu)+k_s-a(\nu_0)\,\phi_s(\nu_0\,|\,\mu) \tag{66}$$

Substituting this expression into Eq. (63), we obtain

$$A(\mu)=\frac{1}{\Omega^+\Omega^-}\left\{\lambda_s(\mu)\, g(\mu)-\frac{1}{X_1(-\mu)}\left[\frac{1}{i\pi}\int_0^1 \frac{1}{2}\frac{(\Omega^+-\Omega^-)\,X_1(-\nu)\,f(\nu)}{\nu-\mu}\,d\nu\right.\right.$$
$$\left.-\frac{1}{i\pi}\int_0^1 \frac{1}{2}\frac{(\Omega^+-\Omega^-)\,X_1(-\nu)\,f(\nu)}{\nu-\nu_0}\,d\nu\right]$$
$$\left.-\frac{a(\nu_0)}{X_1(-\mu)}\,[I_4(\nu_0)-I_3(\mu)]+\frac{k_s}{X_1(-\mu)}\,[I_1(\mu)-I_2(\nu_0)]\right\} \tag{67}$$

where I_1, I_2, I_3, and I_4 are given by the following expressions:

$$I_1(\mu)=\left(\frac{s}{s+1}\right)^2\left(\frac{1}{2\pi i}\int_0^1 \log\frac{\Omega^+}{\Omega^-}\,dv-\mu\right)+\lambda_s(\mu)\,X_1(-\mu) \tag{68a}$$

$$I_2(v_0)=\left(\frac{s}{s+1}\right)^2\left(\frac{1}{2\pi i}\int_0^1 \log\frac{\Omega^+}{\Omega^-}\,dv-v_0\right) \tag{68b}$$

$$I_3(\mu)=-\frac{3}{2}\left(\frac{s}{s+1}\right)^2 v_0^2 I_1(\mu)+\left(1+\frac{3sv_0^2}{s+1}\right)\left[\left(\frac{s}{s+1}\right)^2+\frac{\lambda_s(\mu)\,X_1(-\mu)}{v_0-\mu}\right] \tag{68c}$$

$$I_4(v_0)=-\frac{3}{2}\left(\frac{s}{s+1}\right)^2 v_0^2 I_2(v_0)+\left(1+\frac{3sv_0^2}{s+1}\right)\left(\frac{s}{s+1}\right)^2\left[1+\frac{v_0}{X_1(v_0)}\right] \tag{68d}$$

Multiplying Eq. (67) by $3s\mu^2/(s+1)^2$ and integrating gives a linear equation to determine the unknowns k_s and a; the second equation is obtained from the condition (65):

$$a(v_0)\,I_1'(v_0)-k_s I_2'(v_0)=\int_0^1 \frac{(\Omega^+-\Omega^-)\,X_1(-v)\,f(v)}{v_0^2-v^2}\,dv \tag{69}$$

where

$$I_2'(v_0)=2\pi i\left(\frac{s}{s+1}\right)^2$$

$$I_1'(v_0)=-\frac{3}{2}\left(\frac{s}{s+1}\right) v_0^2 I_2'(v_0)+\left(1+3\frac{s}{s+1}v_0^2\right)\left(\frac{s}{s+1}\right)^2\frac{i\pi}{X_1(v_0)}$$

When $s\in R_2$ the dispersion function has four zeros; hence, we have

$$X_2(z)=(1-z^2)\exp\left(-\frac{1}{2\pi i}\int_0^1 \log\frac{\Omega^+}{\Omega^-}\frac{1}{v-z}\,dv\right) \tag{70}$$

The function $X_2(z)$ is such that

$$\frac{1}{X_2(z)}=\frac{1}{2\pi i}\int_0^1\left(\frac{1}{X_2^+(v)}-\frac{1}{X_2^-(v)}\right)\frac{1}{v-z}\,dv \tag{71a}$$

$$\frac{X_2(z)\,X_2(-z)\,\Omega(z)}{(v_0^2-z^2)(v_1^2-z^2)}=\left(\frac{s}{s+1}\right)^2 \tag{71b}$$

$$\frac{1}{X_2(\mu)}=\left(\frac{s+1}{s}\right)^2\frac{1}{2\pi i}\int_{-1}^0\frac{(\Omega^+-\Omega^-)\,X_2(v)}{(v_0^2-v^2)(v_1^2-v^2)}\frac{1}{v+\mu}\,dv \tag{71c}$$

The coefficient $A(\mu)$ is now given by the following expression:

$$A(\mu)=\frac{1}{\Omega^+\Omega^-}\left\{\lambda_s(\mu)\,g(\mu)\right.$$
$$-\frac{1}{X_2(-\mu)}\left[\frac{v_0+v_1}{2}(v_1-\mu)\frac{1}{i\pi}\int_0^1\frac{(\Omega^+-\Omega^-)\,X_2(-v)\,g(v)}{v-v_0}\,dv\right.$$
$$-\frac{v_0+v_1}{2}(v_0-\mu)\frac{1}{i\pi}\int_0^1\frac{(\Omega^+-\Omega^-)\,X_2(-v)\,g(v)}{v-v_1}$$
$$\left.\left.+\frac{1}{i\pi}\int_0^1\frac{(\Omega^+-\Omega^-)\,X_2(-v)\,g(v)}{v-\mu}\,dv\right]\right\} \tag{72}$$

where the function

$$g(\mu)=f(\mu)+k_s-a(v_0)\,\phi_s(v\,|\,\mu)-a(v_1)\,\phi_s(v_1\,|\,\mu) \tag{73}$$

is such that the two conditions

$$\int_0^1\frac{(\Omega^+-\Omega^-)\,X_2(-v)\,g(v)}{(v_0^2-v^2)(v_1^2-v^2)}\,dv \tag{74a}$$

$$\int_0^1\frac{(\Omega^+-\Omega^-)\,X_2(-v)\,vg(v)}{(v_0^2-v^2)(v_1^2-v^2)}\,dv=0 \tag{74b}$$

are to be satisfied. In deriving Eq. (71), the conditions (74) have been taken into account as well as the identity (71b). The integrals appearing on the right-hand side of Eq. (71) have to be calculated by inserting the expression for $g(\mu)$ given by Eq. (73) and taking into account that

$$\frac{1}{i\pi}\int_0^1\frac{(\Omega^+-\Omega^-)\,X_2(-v)}{v-v_j}=-\left(\frac{s}{s+1}\right)^2(v_l^2T_1-v_jT_2-T_3)$$

$$\frac{1}{i\pi}\int_0^1\frac{(\Omega^+-\Omega^-)\,X_2(-v)}{(v-v_j)^2}\,dv=2\left(\frac{s}{s+1}\right)^2\left[1-\frac{v_j^2}{X_2(v_j)}\right],\qquad j=0,1;\quad l=1,0$$

$$\frac{1}{i\pi}\int_0^1\frac{(\Omega^+-\Omega^-)\,X_2(-v)}{(v-v_0)(v-v_1)}\,dv=\left(\frac{s}{s+1}\right)^2[T_2+(v_0+v_1)T_1]$$

$$\frac{1}{i\pi}\int_0^1\frac{(\Omega^+-\Omega^-)\,X_2(-v)}{v-\mu}\,dv$$
$$=\left(\frac{s}{s+1}\right)^2[T_3+\mu T_2+(\mu^2-v_0^2-v_1^2)T_1]+2\lambda_s(\mu)\,X_2(-\mu)$$

$$\frac{1}{i\pi}\int_0^1 \frac{(\Omega^+ - \Omega^-)\, X_2(-\nu)}{(\nu-\mu)(\nu-\nu_i)}\, d\nu$$

$$= \left(\frac{s}{s+1}\right)^2 (T_2 - 2\mu) + \mu \frac{2\lambda_s(\mu)\, X_2(-\mu)}{\nu_i^2 - \mu^2} - \frac{2\lambda_s(\mu)\, X_2(-\mu)}{(\nu_1^2-\mu^2)(\nu_0^2-\mu^2)}$$

$$T_1 = \frac{1}{i\pi}\int_0^1 \nu \left(\frac{1}{X_2^+(\nu)} - \frac{1}{X_2^-(\nu)}\right) d\nu = -1$$

$$T_2 = \frac{1}{i\pi}\int_0^1 \nu^2 \left(\frac{1}{X_2^+(\nu)} - \frac{1}{X_2^-(\nu)}\right) d\nu = \frac{1}{i\pi}\int_0^1 \log\frac{\Omega^+}{\Omega^-}\, d\nu$$

$$T_3 = \frac{1}{i\pi}\int_0^1 \nu^3 \left(\frac{1}{X_2^+(\nu)} - \frac{1}{X_2^-(\nu)}\right) d\nu = -\frac{1}{i\pi}\int_0^1 \nu \log\frac{\Omega^+}{\Omega^-}\, d\nu - \frac{T_2^2}{4}$$

Inserting the expression (72) into Eqs. (73), we obtain two linear equations containing k_s, a_0, and a_1 as unknowns:

$$\sum_{m=1}^{2} a(\nu_m) I_{1m} = \frac{1}{i\pi}\int_0^1 \frac{(\Omega^+ - \Omega^-)\, X_2(-\nu)\, f(\nu)}{(\nu_0^2-\nu^2)(\nu_1^2-\nu^2)}\, d\nu \qquad (75a)$$

$$-I_2 k_s + \sum_{m=1}^{2} a(\nu_m) I_{2m} = \frac{1}{i\pi}\int_0^1 \frac{(\Omega^+ - \Omega^-)\, X_2(-\nu)\, \nu f(\nu)}{(\nu_0^2-\nu^2)(\nu_1^2-\nu^2)}\, d\nu \qquad (75b)$$

where

$$I_1 = \int_0^1 \frac{(\Omega^+ - \Omega^-)\, X_2(-\nu)}{(\nu_0^2-\nu^2)(\nu_1^2-\nu^2)}\, d\nu = 0$$

$$I_{1m} = \frac{1}{i\pi}\int_0^1 \frac{(\Omega^+ - \Omega^-)\, X_2(-\nu)\, \phi_s(\nu_m \mid \nu)}{(\nu_0^2-\nu^2)(\nu_1^2-\nu^2)}\, d\nu$$

$$= -\frac{\nu_m}{s+1}\left(\frac{s}{s+1}\right)^2 \left(1 + \frac{3s\nu_m^2}{s+1}\right) \frac{1}{X_2(\nu_m)}$$

$$I_2 = \frac{1}{i\pi}\int_0^1 \frac{(\Omega^+ - \Omega^-)\, X_2(-\nu)\, \nu}{(\nu_0^2-\nu^2)(\nu_1^2-\nu^2)}\, d\nu = -2\left(\frac{s}{s+1}\right)^2$$

$$I_{2m} = \frac{1}{i\pi}\int_0^1 \frac{(\Omega^+ - \Omega^-)\, X_2(-\nu)\, \nu\phi_s(\nu_m \mid \nu)}{(\nu_0^2-\nu^2)(\nu_1^2-\nu^2)}\, d\nu$$

$$= \frac{3\nu_m^2 s}{(s+1)^2} - \frac{1}{2}\frac{\nu_m^2}{s+1}\left(1 + \frac{3s\nu_m^2}{s+1}\right)\left(\frac{s}{s+1}\right)^2 \frac{1}{X_2(\nu_m)}$$

The third equation is obtained by multiplying Eq. (72) by $3s\mu^2/(s+1)^2$ and integrating.

4. CONCLUSIONS

The equations of propagation of shear and sound waves in an ultrarelativistic gas have been studied by the method of elementary solutions. The nature of the discrete spectrum associated to each equation has been investigated and the half-space expansion coefficient calculated. It is worth noticing that in the case of the propagation equation for sound waves the calculation of the expansion coefficient requires a considerable amount of numerical evaluation because of some twofold integrals appearing in the coefficients of the linear systems from which k_s is evaluated.

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